THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 I/J University Mathematics 2015-2016
Problem Set 4

1. Evaluate the following limits.

   (a) $\lim_{x \to +\infty} \left(\frac{x+1}{x-1} \right)^x$;

   (b) $\lim_{x \to +\infty} \left(\frac{x^2 - 2x - 3}{x^2 - 3x - 28} \right)^x$;

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \frac{|x-4|}{4-x} & \text{if } x \neq 4; \\ 0 & \text{if } x = 4. \end{cases}$$

   (a) Sketch the graph of the function $f(x)$.

   (b) Is $f(x)$ continuous at $x = 4$? Why?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{e^x - 1}) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

   Show that $f(x)$ is continuous at $x = 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies

   - $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$;
   - $f(x)$ is continuous at $x = 0$ and $f(0) \neq 0$.

   (a) Show that $f(0) = 1$.

   (b) Hence, show that $f(x)$ is continuous on $\mathbb{R}$.

5. (Challenge) Let $f(x)$ be a continuous function defined for $x > 0$ and for any $x, y > 0$,

$$f(xy) = f(x) + f(y).$$

   (a) Find $f(1)$.

   (b) Let a be a positive real number. Prove that for any rational number r,

$$f(a^r) = rf(a).$$

(c) It is known that for all real number x, there exists a sequence $\{x_n\}$ of rational numbers such that $\lim_{n\to\infty} x_n = x$.
Show that for all $x > 0$,

$$f(a^x) = xf(a)$$

where a is a positive real constant. Hence, prove that for all $x > 0$,

$$f(x) = c \ln x$$

where c is a constant.